

Non-perturbative black holes in Type-IIA String Theory vs. the No-Hair conjecture

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Abstract

We obtain the first black hole solution to Type-IIA String Theory compactified on an arbitrary self-mirror Calabi Yau manifold in the presence of non-perturbative quantum corrections. Remarkably enough, the solution involves multivalued functions, which could lead to a violation of the No-Hair conjecture. We discuss how String Theory forbids such scenario. However the possibility still remains open in the context of four-dimensional ungauged Supergravity.

Résumé

Nous avons obtenu la toute première solution de trou noir dans le cadre d'une théorie des cordes de type IIA compactifiée sur une variété de Calabi-Yau auto-miroir arbitraire, en présence de corrections quantiques non-perturbatives. De manière remarquable, cette solution fait intervenir des fonctions multivaluées, ce qui pourrait entraîner une violation de la conjecture de calvitie du trou noir. Nous expliquons en quoi tel scénario est interdit en théorie des cordes, mais reste a priori possible en Supergravité quadridimensionnelle non jaugée.

Introduction

Black hole physics is an extremely active research field in String Theory. Some impressive results have been obtained towards a complete match of the microscopic and the macroscopic entropies of extremal black holes [1] beyond leading order, see [2, 3] and references therein. So far, most of the literature has been focused on the so called *higher-order* (curvature) corrections [4–8]. These modify the usual Bekenstein-Hawking area law

$$S = \frac{A}{4},$$

and are prescribed by String Theory to appear in the effective classical Supergravity description.

In the context of Type-IIA, the effective theory is described by Supergravity at the classical level (at tree level in g_s), with the higher-order corrections occurring already at the 1-loop level.

There exist, however, a different type of *stringy* corrections (which correct the point particle behaviour), some of which do not modify the effective Lagrangian of Type-IIA String Theory with higher order terms, but modify the couplings and the scalar manifold geometry of the classical Supergravity action. These are the α' -corrections¹, often referred to as *quantum* corrections² [9–12]. In contradistinction to the curvature ones, these do not modify the Bekenstein-Hawking area law³.

It turns out that not so much attention has been paid to the effects of these *quantum* corrections to black hole solutions of the classical Supergravity action. Some exceptions, in which this kind of solutions in the presence of these have been considered, are [13–17].

In [18], using the H-FGK formalism⁴ [21–24], a new class of black holes for Type-IIA Calabi-Yau compactifications was defined: they exist only when the perturbative corrections to the prepotential are included and no classical limit can be assigned to them. They were

called, in consequence, *quantum* black holes. Therefore, for self-mirror Calabi-Yau manifolds such black holes do not exist, since in such case the perturbative corrections exactly vanish. However, the situation can be changed if we add non-perturbative corrections to the prepotential. That is the case we are going to consider in this letter. We will obtain the first explicit black hole solution of Type-IIA String Theory compactified on a self-mirror Calabi-Yau threefold in the presence of non-perturbative corrections, proving at the same time that these non-perturbative corrections *lift* the singular behaviour of the quantum black holes to a regular one.

The solution (general class of solutions, in fact) may allow also for an explicit check of the match between the microscopic String Theory entropy and the macroscopic entropy of a supersymmetric black hole solution in the presence of these non-perturbative corrections⁵. To that respect, some partial results can be found in [13]. In any case, computing the macroscopic solution is undoubtedly the first step towards the resolution of this puzzle.

Somewhat surprisingly, we obtain a class of solutions which involves Lambert's W function [25], which is multi-valued in a certain real domain. We will explain how this fact seems to provide an appropriate scenario for a potential violation of the (corresponding uniqueness) conjecture, and as a consequence of the) No-Hair conjecture in four dimensions. It turns out that, in our set-up, String Theory forbids the use of the Lambert function to that end. However, the possibility remains open in an exclusive Supergravity set-up (not necessarily embedded in String Theory) [26], and there does not seem to be a reason to discard it right away. Black holes in ungauged four-dimensional Supergravity still hold some surprises [27, 28], which indicate that we may not completely understand them even in the extremal case.

1 Type-IIA String Theory on a Calabi–Yau manifold

Type-IIA String Theory compactified to 4D on a C.Y. three-fold, with Hodge numbers $(h^{1,1}, h^{2,1})$, is described, up to two derivatives, by a $\mathcal{N} = 2, d = 4$ Supergravity whose prepotential is given in terms

¹Perturbative (world-sheet loops) and non-perturbative (worldsheet instantons).

²At least in the context of Type-IIA C.Y. compactifications.

³They obviously modify the area of the black hole itself, as far as the structure of the solution is changed when they are taken into account.

⁴A related formalism can be found in [19, 20].

⁵This has not been achieved yet and nowadays seems to be out of reach.

of an infinite series around $\Im m z^i \rightarrow \infty^6$ [9, 10, 29]

$$\mathcal{F} = -\frac{1}{3!} \kappa_{ijk}^0 z^i z^j z^k + \frac{ic}{2} + \frac{i}{(2\pi)^3} \sum_{\{d_i\}} n_{\{d_i\}} \text{Li}_3 \left(e^{2\pi i d_i z^i} \right) , \quad (1.1)$$

where z^i , $i = 1, \dots, n_v + 1 = h^{1,1}$, are the scalars in the vector multiplets, $\frac{7}{c} = \frac{\chi\zeta(3)}{(2\pi)^3}$ is a model-dependent number⁸, κ_{ijk}^0 are the classical intersection numbers, $d_i \in \mathbb{Z}^+$ is a $h^{1,1}$ -dimensional summation index and $\text{Li}_3(x)$ is the third polylogarithmic function, defined in appendix A. The first two terms in the prepotential correspond to tree level and loop perturbative contributions in the α' -expansion, respectively

$$\mathcal{F}_P = -\frac{1}{3!} \kappa_{ijk}^0 z^i z^j z^k + \frac{ic}{2} , \quad (1.2)$$

whereas the third term accounts for non-perturbative corrections produced by world-sheet instantons. These configurations get produced by (non-trivial) embeddings of the world-sheet into the C.Y. three-fold. The holomorphic mappings of the genus 0⁹ string world-sheet onto the $h^{1,1}$ two-cycles of the C.Y. three-fold are classified by the numbers d_i , which count the number of wrappings of the world-sheet around the i -th generator of the integer homology group $H_2(\text{C.Y.}, \mathbb{Z})$. The number of different mappings for each set of $\{d_i\}$ ($\equiv \{d_1, \dots, d_{h^{1,1}}\}$) or, in other words, the number of genus 0 instantons is denoted by $n_{\{d_i\}}$ ¹⁰

$$\mathcal{F}_{NP} = \frac{i}{(2\pi)^3} \sum_{\{d_i\}} n_{\{d_i\}} \text{Li}_3 \left(e^{2\pi i d_i z^i} \right) . \quad (1.3)$$

The full prepotential can be rewritten in homogeneous coordinates \mathcal{X}^Λ , $\Lambda = (0, i)$ as

$$\begin{aligned} F(\mathcal{X}) = & -\frac{1}{3!} \kappa_{ijk}^0 \frac{\mathcal{X}^i \mathcal{X}^j \mathcal{X}^k}{\mathcal{X}^0} + \frac{ic(\mathcal{X}^0)^2}{2} \\ & + \frac{i(\mathcal{X}^0)^2}{(2\pi)^3} \sum_{\{d_i\}} n_{\{d_i\}} \text{Li}_3 \left(e^{2\pi i d_i \frac{\mathcal{X}^i}{\mathcal{X}^0}} \right) , \end{aligned} \quad (1.4)$$

with the scalars z^i given by¹¹

$$z^i = \frac{\mathcal{X}^i}{\mathcal{X}^0} . \quad (1.5)$$

We are interested in studying spherically symmetric, static, black hole solutions of the theory defined by Eq. (1.1). In order to do so we are going to use the so-called H-FGK, developed in [21–23], based on the use of a new set of variables H^M , $M = (\Lambda, \Lambda)$, which transform linearly under duality and reduce to harmonic functions on the transverse space \mathbb{R}^3 in the supersymmetric case.

⁶Actually, the prepotential obtained in a Type-IIA C.Y. compactification is *symplectically equivalent* to the prepotential (1.1).

⁷There are also $h^{2,1} + 1$ hypermultiplets in the theory. However, they can be consistently set to a constant value.

⁸ χ is the Euler characteristic, which for C.Y. three-folds is given by $\chi = 2(h^{1,1} - h^{2,1})$.

⁹Genus ≥ 1 instantons contribute with higher-derivative corrections.

¹⁰See, e.g. [7] for more details on the stringy origin of the prepotential.

¹¹This coordinate system is therefore only valid away from the locus $\mathcal{X}^0 = 0$.

2 A non-perturbative class of black holes

The most general static, spherically symmetric space-time metric solution of an ungauged Supergravity is given by¹² [22, 30]

$$\begin{aligned} ds^2 &= e^{2U(\tau)} dt^2 - e^{-2U(\tau)} \gamma_{mn} dx^m dx^n , \\ \gamma_{mn} dx^m dx^n &= \frac{r_0^4}{\sinh^4 r_0 \tau} d\tau^2 + \frac{r_0^2}{\sinh^2 r_0 \tau} d\Omega_{(2)}^2 . \end{aligned} \quad (2.1)$$

Using Eq. (2.1) and following the H-FGK formalism, we obtain that the equations of the theory are given by

$$\begin{aligned} \mathcal{E}_P &= \frac{1}{2} \partial_P \partial_M \partial_N \log W \left[\dot{H}^M \dot{H}^N - \frac{1}{2} \mathcal{Q}^M \mathcal{Q}^N \right] \\ &+ \partial_P \partial_M \log W \ddot{H}^M - \frac{d}{d\tau} \left(\frac{\partial \Lambda}{\partial \dot{H}^P} \right) + \frac{\partial \Lambda}{\partial H^P} = 0 , \end{aligned} \quad (2.2)$$

together with the *Hamiltonian constraint*

$$\begin{aligned} \mathcal{H} &\equiv -\frac{1}{2} \partial_M \partial_N \log W \left(\dot{H}^M \dot{H}^N - \frac{1}{2} \mathcal{Q}^M \mathcal{Q}^N \right) \\ &+ \left(\frac{\dot{H}^M H_M}{W} \right)^2 - \left(\frac{\mathcal{Q}^M H_M}{W} \right)^2 - r_0^2 = 0 , \end{aligned} \quad (2.3)$$

where

$$\Lambda \equiv \left(\frac{\dot{H}^M H_M}{W} \right)^2 + \left(\frac{\mathcal{Q}^M H_M}{W} \right)^2 , \quad (2.4)$$

and

$$W(H) \equiv \tilde{H}_M(H) H^M = e^{-2U}, \quad \tilde{H}^M + i\mathcal{I}^M = \mathcal{V}^M/X . \quad (2.5)$$

\mathcal{V}^M is the covariantly holomorphic symplectic section of $\mathcal{N} = 2$ Supergravity, and X is a complex variable with the same Kähler weight as \mathcal{V}^M . $\tilde{H}^M(\mathcal{I}) \equiv \tilde{H}^M(H)$ stands for the real part (\tilde{H}^M) of \mathcal{V}^M written as a function of the imaginary part $\mathcal{I}^M \equiv H^M$, something that can always be done by solving the so-called *stabilization equations*, which we will explicitly write down in a moment. $W(H)$ is usually known in the literature as the *Hesse potential*.

The effective theory is now expressed in terms of $2(n_v + 1)$ variables H^M and depends on $2(n_v + 1) + 1$ parameters: $2(n_v + 1)$ charges \mathcal{Q}^M and the non-extremality parameter r_0 , from which it is possible to reconstruct the solution in terms of the original fields of the theory (that is it, the space-time metric, scalars and vector fields).

In order to tackle the construction of new black hole solutions of (1.1), we are going to consider a particular consistent truncation given by

$$H^0 = H_0 = H_i = 0, \quad p^0 = q_0 = q_i = 0 . \quad (2.6)$$

Under this assumption, the stabilization equations, which can be directly read off from (2.5) take the form

$$\begin{pmatrix} iH^i \\ \tilde{H}_i \end{pmatrix} = \frac{e^{\mathcal{K}/2}}{X} \begin{pmatrix} \mathcal{X}^i \\ \frac{\partial F(\mathcal{X})}{\partial \mathcal{X}^i} \end{pmatrix}, \quad \begin{pmatrix} \tilde{H}^0 \\ 0 \end{pmatrix} = \frac{e^{\mathcal{K}/2}}{X} \begin{pmatrix} \mathcal{X}^0 \\ \frac{\partial F(\mathcal{X})}{\partial \mathcal{X}^0} \end{pmatrix}, \quad (2.7)$$

and the physical fields can be obtained from the H^i as

$$e^{-2U} = \tilde{H}_i H^i, \quad z^i = i \frac{H^i}{\tilde{H}^0}, \quad (2.8)$$

¹²The conformastatic coordinates (t, τ, θ, ϕ) cover the outer region of the event horizon when $\tau \in (-\infty, 0)$ and the inner region, between the Cauchy horizon and the physical singularity when $\tau \in (\tau_S, \infty)$, where $\tau_S \in \mathbb{R}^+$ is a model dependent number. The event horizon is located at $\tau \rightarrow -\infty$ and the Cauchy horizon at $\tau \rightarrow \infty$.

as soon as \tilde{H}^0 and \tilde{H}^i are determined. In order to obtain \tilde{H}^0 as a function of H^i , we need to solve the highly involved equation

$$\frac{\partial F(H)}{\partial \tilde{H}^0} = 0, \quad (2.9)$$

where $F(H)$ stands for the prepotential expressed in terms of the H^i

$$\begin{aligned} F(H) = & \frac{i}{3!} \kappa_{ijk}^0 \frac{H^i H^j H^k}{\tilde{H}^0} + \frac{i c (\tilde{H}^0)^2}{2} \\ & + \frac{i (\tilde{H}^0)^2}{(2\pi)^3} \sum_{\{d_i\}} n_{\{d_i\}} \text{Li}_3 \left(e^{-2\pi d_i \frac{H^i}{\tilde{H}^0}} \right). \end{aligned} \quad (2.10)$$

Once this is done, it is not difficult to express \tilde{H}^i in terms of H^i . Indeed, from (2.7) we simply have

$$\tilde{H}_i = -i \frac{\partial F(H)}{\partial H^i}. \quad (2.11)$$

If we expand (2.9), we find

$$\begin{aligned} -\frac{1}{3!} \kappa_{ijk}^0 \frac{H^i H^j H^k}{(\tilde{H}^0)^3} + c + \frac{1}{4\pi^3} \sum_{\{d_i\}} n_{\{d_i\}} \left[\text{Li}_3 \left(e^{-2\pi d_i \frac{H^i}{\tilde{H}^0}} \right) \right. \\ \left. + \text{Li}_2 \left(e^{-2\pi d_i \frac{H^i}{\tilde{H}^0}} \right) \left[\frac{\pi d_i H^i}{\tilde{H}^0} \right] \right] = 0. \end{aligned} \quad (2.12)$$

Solving (2.12) for \tilde{H}^0 in full generality seems to be an extremely difficult task. However, if we go to the large volume compactification limit ($\Im m z^i \gg 1$), we can make use of the following property of the polylogarithmic functions

$$\lim_{|w| \rightarrow 0} \text{Li}_s(w) = w, \forall s \in \mathbb{N}, \quad (2.13)$$

since, in our case, $w = e^{-2\pi d_i \Im m z^i}$, $\forall \{d_i\} \in (\mathbb{Z}^+)^{h^{1,1}}$. Eq. (2.13) enables us to rewrite (2.12) as

$$\begin{aligned} -\frac{1}{3!} \kappa_{ijk}^0 \frac{H^i H^j H^k}{(\tilde{H}^0)^3} + c + \frac{1}{4\pi^3} \sum_{\{d_i\}} n_{\{d_i\}} \left[e^{-2\pi d_i \frac{H^i}{\tilde{H}^0}} \right. \\ \left. + e^{-2\pi d_i \frac{H^i}{\tilde{H}^0}} \left[\frac{\pi d_i H^i}{\tilde{H}^0} \right] \right] = 0, \quad \Im m z^i \gg 1. \end{aligned} \quad (2.14)$$

The dominant contribution in this regime, aside from the cubic one, is given by c . In [18] [17], the first non-extremal black hole solutions (with constant and non-constant scalars) of (1.1) were obtained ignoring the non-perturbative corrections. In particular, the solutions of [18] turned out to be purely *quantum*¹³, in the sense that not only the classical limit $c \rightarrow 0$ was ill-defined, but also the truncated theory became inconsistent and therefore no classical limit could be assigned to such solutions. An interesting question to ask now is whether the non-perturbative contributions could actually be able to cure or at least improve this behaviour. On the other hand, it is also interesting *per se* to explore the existence of black hole solutions when the subleading contribution to the prepotential is not given by c , but has a non-perturbative origin. In order to tackle these two questions, let us restrict ourselves to C.Y. three-folds with vanishing Euler characteristic ($c = 0$), the so-called self-mirror C.Y. three-folds.

¹³It is worth pointing out again that the term *quantum* does not refer to space-time but to world-sheet properties in this context [7]. In this respect, although such denomination is widely spread in the literature, the adjective *stringy* might result more accurate.

Under this assumption, and considering only the subleading contribution in (2.12), which is now given by the fourth term in (2.14), such equation becomes¹⁴

$$-\frac{1}{3!} \kappa_{ijk}^0 \frac{H^i H^j H^k}{(\tilde{H}^0)^3} + \frac{1}{4\pi^3} \sum_{\{d_i\}} n_{\{d_i\}} e^{-2\pi d_i \frac{H^i}{\tilde{H}^0}} \left[\frac{\pi d_i H^i}{\tilde{H}^0} \right] = 0. \quad (2.15)$$

The sum over $\{d_i\}$ in (2.15) will be dominated in each case by a certain term corresponding to a particular vector $\{\hat{d}_i\}$ (and, as a consequence, to a particular $n_{\hat{d}_i} \equiv \hat{n}$), which, since we are assuming $\Im m z^i \gg 1$, is the only one that we need to consider. Therefore, (2.15) becomes

$$-\frac{1}{3!} \kappa_{ijk}^0 \frac{H^i H^j H^k}{(\tilde{H}^0)^3} + \frac{\hat{n}}{4\pi^3} e^{-2\pi \hat{d}_i \frac{H^i}{\tilde{H}^0}} \left[\frac{\pi \hat{d}_i H^i}{\tilde{H}^0} \right] = 0, \quad \Im m z^i \gg 1. \quad (2.16)$$

This equation is solved by¹⁵

$$\tilde{H}^0 = \frac{\pi \hat{d}_i H^i}{W_a \left(s_a \sqrt{\frac{3\hat{n}(\hat{d}_i H^i)^3}{2\kappa_{ijk}^0 H^i H^j H^k}} \right)}, \quad (2.17)$$

where $W_a(x)$, ($a = 0, -1$) stands for (any of the two real branches of) the *Lambert W function*¹⁶ (also known as *product logarithm*), and $s_a = \pm 1$. Using now Eqs. (2.17) and (2.11) we can obtain \tilde{H}^i . The result is

$$\tilde{H}_i = \frac{1}{2} \kappa_{ijk}^0 \frac{H^j H^k}{\pi \hat{d}_i H^i} W_a \left(s_a \sqrt{\frac{3\hat{n}(\hat{d}_i H^i)^3}{2\kappa_{pqr}^0 H^p H^q H^r}} \right). \quad (2.18)$$

The physical fields can now be written as a function of the H^i as

$$e^{-2U} = \mathbb{W}(H) = \frac{\kappa_{ijk}^0 H^i H^j H^k}{2\pi \hat{d}_m H^m} W_a \left(s_a \sqrt{\frac{3\hat{n}(\hat{d}_i H^i)^3}{2\kappa_{pqr}^0 H^p H^q H^r}} \right), \quad (2.19)$$

$$z^i = i \frac{H^i}{\pi \hat{d}_m H^m} W_a \left(s_a \sqrt{\frac{3\hat{n}(\hat{d}_i H^i)^3}{2\kappa_{pqr}^0 H^p H^q H^r}} \right). \quad (2.20)$$

In order to have a regular solution, we need to have a positive definite metric warp factor e^{-2U} . Since, as explained in Appendix B, $\text{sign}[W_a(x)] = \text{sign}[x]$, $a = 0, -1$, $x \in D_{\mathbb{R}}^a$, we have to require that

$$s_0 \equiv \text{sign} \left[\kappa_{ijk}^0 \frac{H^i H^j H^k}{\hat{d}_m H^m} \right], \quad (2.21)$$

$$s_{-1} \equiv -1. \quad (2.22)$$

On the other hand, since $W_0(x) = 0$ for $x = 0$ and $W_{-1}(x)$ is a real function only when $x \in [-\frac{1}{e}, 0)$, we have to impose that the argument x of W_a lies entirely either in $[-\frac{1}{e}, 0)$ or in $(0, \infty)$ for all $\tau \in (-\infty, 0)$, since e^{2U} cannot be zero in a regular black hole solution for any $\tau \in (-\infty, 0)$. This condition must be imposed in a case by case basis, since it depends on the specific form of the symplectic vector $H^M = H^M(\tau)$ as a function of τ . Notice that if

¹⁴ $e^{2\pi i d_i z^i} \ll \pi d_i \Im m z^i e^{2\pi i d_i z^i}$ for $\Im m z^i \gg 1$.

¹⁵Henceforth we will be using \mathbb{W} for the Hessian potential, and W for the Lambert function. We hope this is not a source of confusion.

¹⁶See the Appendix B for more details.

$x \in [-\frac{1}{e}, 0)$ $\forall \tau \in (-\infty, 0)$ we can in principle¹⁷ choose either W_0 or W_{-1} to build the solution, whereas if $x \in (0, +\infty)$ $\forall \tau \in (-\infty, 0)$, only W_0 is available.

Needless to say, in order to construct actual solutions, we have to solve the H-FGK equations of motion (2.2) (plus hamiltonian constraint (2.3)) using the Hessian potential given by (2.19). Fortunately, such equations admit a *model-independent* solution which is obtained choosing the H^i to be harmonic functions in the flat transverse space, with one of the poles given in terms of the corresponding charge

$$H^i = a^i - \frac{p^i}{\sqrt{2}}\tau, \quad r_0 = 0. \quad (2.23)$$

This corresponds to a supersymmetric black hole [31–33].

3 The general supersymmetric solution

As we have said, plugging (2.23) into (2.20) and (2.19) provides us with a supersymmetric solution without solving any further equation. The entropy of such a solution reads

$$S = \frac{1}{2} \kappa_{ijk}^0 \frac{p^i p^j p^k}{\hat{d}_m p^m} W_a(s_a \alpha), \quad (3.1)$$

$$\beta = \sqrt{\frac{3\hat{n}(\hat{d}_l p^l)^3}{2\kappa_{pqr}^0 p^p p^q p^r}},$$

and the mass is given by

$$M = \dot{U}(0) = \frac{1}{2\sqrt{2}} \left[\frac{3\kappa_{ijk}^0 p^i a^j a^k}{\kappa_{pqr}^0 a^p a^q a^r} \left[1 - \frac{1}{1 + W_a(s_a \alpha)} \right] - \frac{d_l p^l}{d_n a^n} \left[1 - \frac{3}{2(1 + W_a(s_a \alpha))} \right] \right], \quad (3.2)$$

$$\alpha = \sqrt{\frac{3\hat{n}(d_l a^l)^3}{2\kappa_{pqr}^0 a^p a^q a^r}}. \quad (3.3)$$

In the approximation under consideration, we are neglecting terms $\sim e^{-2\pi d_i \Im m z^i}$ with respect to those going as $\sim \pi d_i \Im m z^i e^{-2\pi d_i \Im m z^i}$. Taking into account (2.20), this assumption is translated into the condition

$$W_a(x) e^{-2W_a(x)} \gg e^{-2W_a(x)}. \quad (3.4)$$

It is clear that this condition is satisfied for $a = 0$ if $x \in [\alpha, \beta]$ for positive and sufficiently large values of α and β . However, it is not satisfied at all for $x \in [-\frac{1}{e}, 0)$, which is the range for which both branches of the Lambert function are available.

If we assume $x \in [\alpha, \beta]$ for sufficiently large $\alpha, \beta \in \mathbb{R}^+$, $a = 0$ and W_0 is the only real branch of the Lambert function. In that case, $s = s_0 = 1$, and we have

$$e^{-2U} = \frac{\kappa_{ijk}^0 H^i H^j H^k}{2\pi \hat{d}_m H^m} W_0 \left(\sqrt{\frac{3\hat{n}(\hat{d}_l H^l)^3}{2\kappa_{pqr}^0 H^p H^q H^r}} \right), \quad (3.5)$$

$$z^i = i \frac{H^i}{\pi \hat{d}_m H^m} W_0 \left(\sqrt{\frac{3\hat{n}(\hat{d}_l H^l)^3}{2\kappa_{pqr}^0 H^p H^q H^r}} \right). \quad (3.6)$$

In the conformastatic coordinates we are working with, the metric warp factor e^{-2U} is expected to diverge at the event horizon

¹⁷As we will see in section 3, the possibility $s_0 = s_{-1} = -1$ will not be consistent with the large volume approximation we are considering.

$(\tau \rightarrow -\infty)$ as τ^2 . In addition, we have to require $e^{-2U} > 0 \forall \tau \in (-\infty, 0]$, and impose asymptotic flatness $e^{-2U(\tau=0)} = 1$. The last two conditions read

$$\frac{\kappa_{ijk}^0 H^i H^j H^k}{2\pi \hat{d}_n H^n} > 0 \quad \forall \tau \in (-\infty, 0], \quad (3.7)$$

$$\frac{\kappa_{ijk}^0 a^i a^j a^k}{2\pi \hat{d}_m a^m} W_0(\alpha) = 1, \quad (3.8)$$

whereas the first one turns out to hold, since

$$e^{-2U} \xrightarrow{\tau \rightarrow -\infty} \frac{\kappa_{ijk}^0 p^i p^j p^k}{8\pi \hat{d}_m p^m} W_0(\beta) \tau^2. \quad (3.9)$$

(3.7) and (3.8) can in general be safely imposed in any particular model we consider. Finally, the condition for a well-defined and positive mass $M > 0$ can be read off from (3.2).

4 Multivalued functions and the No-Hair conjecture

As we explained in the previous section, our approximation is not consistent with a solution such that $x \in [-\frac{1}{e}, 0)$. This forbids the domain in which $W(x)$ is a multivalued function (both W_0 and W_{-1} are real there). However, it seems legitimate to ask what the consequences of having two different branches would have been, had this constraint not been present. In principle, we could have tried to assign the asymptotic ($\tau \rightarrow 0$) and near horizon ($\tau \rightarrow -\infty$) limits to any particular pair of values of the arguments of W_0 and W_{-1} (x_0 and x_{-1} respectively) through a suitable election of the parameters available in the solution. In particular, had we chosen $x_0|_{\tau=0} = x_{-1}|_{\tau=0} = -1/e$ and $x_0|_{\tau \rightarrow -\infty} = x_{-1}|_{\tau \rightarrow -\infty} = \beta$, $\beta \in (-1/e, 0)$, both solutions would have had exactly the same asymptotic behavior (and therefore the scalars of both solutions would have coincided at spatial infinity), and we would have been dealing with two completely different regular solutions with the same mass¹⁸, charges and asymptotic values of the scalar fields, in flagrant contradiction¹⁹ with the corresponding black hole uniqueness conjecture (and, as a consequence, with the No-Hair conjecture). At this point, and provided that our approximation is not consistent with such presumable two-branched solution, the feasibility of this reasoning in a different context can only be catalogued as *speculative* at the very least. However, a violation of the No-Hair conjecture in four dimensions would have far-reaching consequences independently of whether the solution is embedded in String Theory or not. In this regard, the very possibility that the stabilization equations may admit (for certain more or less complicated prepotentials) solutions depending on multivalued functions seems to open up a window for possible violations of the No-Hair conjecture in the context of $\mathcal{N} = 2$ $d = 4$ Supergravity. The question (whose answer is widely assumed to be "no") is now: is it possible to find a four-dimensional (Super)gravity theory with a physically-admissible matter content admitting more than one stable black hole solution with the same mass, electric, magnetic and scalar charges? If not, why? These questions will be addressed in a forthcoming publication [26].

¹⁸Although $W'_{0,-1}(x)$ are divergent at $x = -1/e$ (as explained in the Appendix B), and the definition of M would involve derivatives of the Lambert function at that point, it would not be difficult to cure this behaviour and get a positive (and finite) mass by imposing $\dot{x}(\tau) \xrightarrow{\tau \rightarrow 0} 0$ faster than $|W'_{0,-1}(x)| \xrightarrow{x \rightarrow -1/e} \infty$.

¹⁹Up to possible stability issues, which should be carefully studied.

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A The polylogarithm

The *polylogarithmic function* or *polylogarithm* $Li_w(z)$ (see e.g. [34] for an exhaustive study) is a special function defined through the power series

$$Li_w(z) = \sum_{j=1}^{\infty} \frac{z^j}{j^w}, \quad z, w \in \mathbb{C}. \quad (\text{A.1})$$

This definition is valid for arbitrary complex numbers w and z for $|z| < 1$, but can be extended to z 's with $|z| \geq 1$ by analytic continuation. From its definition, it is easy to find the recurrence relation

$$Li_{w-1}(z) = z \frac{\partial Li_w(z)}{\partial z}. \quad (\text{A.2})$$

The case $w = 1$ corresponds to

$$Li_1(z) = -\log(1-z), \quad (\text{A.3})$$

and from this it is easy to see that for $w = -n \in \mathbb{Z}^- \cup \{0\}$, the polylogarithm is an elementary function given by

$$Li_0(z) = \frac{z}{1-z}, \quad Li_{-n}(z) = \left(z \frac{\partial}{\partial z}\right)^n \frac{z}{1-z}. \quad (\text{A.4})$$

The special cases $w = 2, 3$ are called *dilogarithm* and *trilogarithm* respectively, and their integral representations can be obtained from $Li_1(z)$ making use of

$$Li_w(z) = \int_0^z \frac{Li_{w-1}(s)}{s} ds. \quad (\text{A.5})$$

B The Lambert W function

The *Lambert W function* $W(z)$ (also known as *product logarithm*) is named after Johann Heinrich Lambert (1728–1777), who was the first to introduce it in 1758 [25]. During its more than two hundred years of history, it has found numerous applications in different areas of physics (mainly during the 20th century) such as electrostatics, thermodynamics (e.g. [35]), statistical physics (e.g. [36]), QCD (e.g. [37–41]), cosmology (e.g. [42]), quantum mechanics (e.g. [43]) and general relativity (e.g. [44]).

$W(z)$ is defined implicitly through the equation

$$z = W(z)e^{W(z)}, \quad \forall z \in \mathbb{C}. \quad (\text{B.1})$$

Since $f(z) = ze^z$ is not an injective mapping, $W(z)$ is not uniquely defined, and $W(z)$ generically stands for the whole set of branches solving (B.1). For $W : \mathbb{R} \rightarrow \mathbb{R}$, $W(x)$ has two branches $W_0(x)$ and $W_{-1}(x)$ defined in the intervals $x \in [-1/e, +\infty)$ and $x \in [-1/e, 0)$ respectively (See Figure 1). Both functions coincide in the

branching point $x = -1/e$, where $W_0(-1/e) = W_{-1}(-1/e) = -1$. As a consequence, the defining equation $x = W(x)e^{W(x)}$ admits two different solutions in the interval $x \in [-1/e, 0)$.

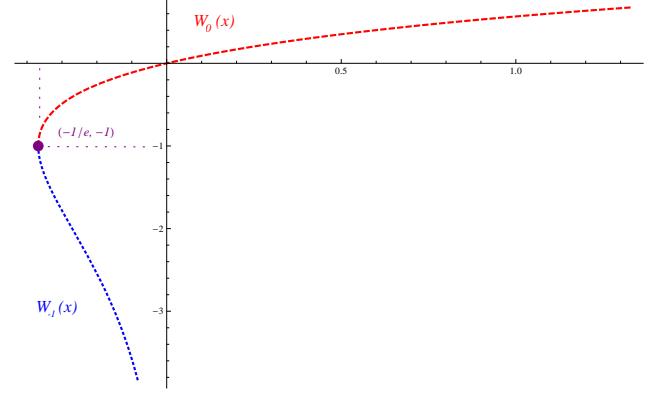


Figure 1: The two real branches of $W(x)$.

The derivative of $W(z)$ reads

$$\frac{dW(z)}{dz} = \frac{W(z)}{z(1+W(z))}, \quad \forall z \notin \{0, -1/e\}; \quad \left. \frac{dW(z)}{dz} \right|_{z=0} = 1, \quad (\text{B.2})$$

and is not defined for $z = -1/e$ (the function is not differentiable there). At that point we have

$$\lim_{x \rightarrow -1/e} \frac{dW_0(x)}{dx} = \infty, \quad \lim_{x \rightarrow -1/e} \frac{dW_{-1}(x)}{dx} = -\infty. \quad (\text{B.3})$$

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